Problem 13

Suppose f is a function that satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x\to 0} \frac{f(x)}{x} = 1$$

(a) Find $f(0)$. (b) Find $f'(0)$. (c) Find $f'(x)$.

Solution

The function f is arbitrary and satisfies

$$f(x+y) = f(x) + x^2y + xy^2 + f(y).$$

Recognize that this is similar to a cubic function.

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$\frac{1}{3}(x+y)^3 = \frac{1}{3}x^3 + x^2y + xy^2 + \frac{1}{3}y^3$$

Therefore,

$$f(x) = \frac{1}{3}x^3 + g(x),$$

where $g(x)$ is another arbitrary function that satisfies

$$g(x+y) = g(x) + g(y).$$

This condition on g is characteristic of a linear function. So then

$$f(x) = \frac{1}{3}x^3 + Ax + B,$$

where A and B are arbitrary constants. Find them using the given limit.

$$1 = \lim_{x\to 0} \frac{f(x)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3}x^3 + Ax + B}{x}$$

In order for this limit to exist and be equal to 1, set $B = 0$ so that the factor of x in the denominator cancels out.

$$1 = \lim_{x\to 0} \frac{\frac{1}{3}x^3 + Ax}{x}$$

$$= \lim_{x\to 0} \left(\frac{1}{3}x^2 + A\right)$$

Use the appropriate limit laws to evaluate the limit.

$$\begin{aligned} 1 &= \lim_{x \to 0} \frac{1}{3}x^2 + \lim_{x \to 0} A \\ &= \frac{1}{3}(0)^2 + A \\ &= A \end{aligned}$$

Consequently, with $A = 1$ and $B = 0$,

$$f(x) = \frac{1}{3}x^3 + x,$$

which means

$$f(0) = \frac{1}{3}(0)^3 + (0) = 0$$

and

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\left[\frac{1}{3}(x+h)^3 + (x+h)\right] - \left[\frac{1}{3}x^3 + x\right]}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{3}(x^3 + 3x^2h + 3xh^2 + h^3) + x + h - \frac{1}{3}x^3 - x}{h} \\ &= \lim_{h \to 0} \frac{x^2h + xh^2 + \frac{h^3}{3} + h}{h} \\ &= \lim_{h \to 0} \left(x^2 + xh + \frac{h^2}{3} + 1\right) \\ &= x^2 + 1 \end{aligned}$$

and

$$f'(0) = (0)^2 + 1 = 1.$$